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SOME NEW STRONGLY REGULAR GRAPHS

A.E. BROUWER, A.V. IVANOV and M.H. KLIN

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We show that three pairwise 4-regular graphs constructed by the second author are members of infinite families.

1. Introduction

A graph Γ is called pairwise k-regular when for each graph K on k vertices with designated pair of vertices $(\varkappa_1, \varkappa_2)$, the number of embeddings of K as induced subgraph of Γ such that $(\varkappa_1, \varkappa_2)$ is mapped to (γ_1, γ_2) does not depend on the vertices γ_1 and γ_2 of Γ but only on their relation (equal, adjacent, or nonadjacent). [Note that other definitions for the concept of pairwise k-regularity occur in the literature.]

Clearly, pairwise k-regularity implies pairwise $(k-1)$-regularity (for $k \geqq 2$), and pairwise 3-regularity is equivalent to strong regularity. A rank 3 graph (i.e., a graph with automorphism group transitive on ordered edges and nonedges) is pairwise k-regular for all $k \geqq 1$.

HIGMAN and HESTENES [6] introduced pairwise k-regularity under the name 'k-vertex condition'; they showed that pairwise 4-regularity for a graph Γ is equivalent to the existence of constants α and β such that for each edge (resp. nonedge) xy there are α (resp. β) edges in the subgraph $\Gamma(x) \cap \Gamma(y)$ induced by the set of common neighbours of x and y in Γ, and that these constants are related by the equiation

$$k\left(\binom{\lambda}{2}-\alpha\right) = (v-k-1)\beta.$$

Clearly, when both subconstituents of a graph are strongly regular (i.e., when it is 'triplewise 4-regular'), then it is pairwise 4-regular.

The second author constructed three strongly regular graphs that are pairwise 4-regular but not rank 3. The parameters are given below.

	v	k	λ	μ	r	s	α	β	$\|G\|$	Remarks
Γ_0	256	120	56	56	8	-8	784	672	$2^{20}\cdot 3^2\cdot 5\cdot 7$	*rank* 4: $1+120+120+15$
Γ_1	120	56	28	24	8	-4	216	144	$2^{12}\cdot 3^2\cdot 5\cdot 7$	*rank* 4: $1+56+56+7$
Γ_2	135	64	28	32	4	-8	168	192	$2^{12}\cdot 3^2\cdot 5\cdot 7$	*intransitive*: $120+15$

These graphs were found by computer; Γ_1 and Γ_2 are the subconstituents of Γ_0.

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The third author gave a direct construction for Γ_1 which is equivalent to the following: Γ_1 is the graph with as vertices one orbit of 8-cocliques under 2^6. Alt (8) in the folded halved 8-cube, where two 8-cocliques are adjacent when they have two points in common.

In this note we shall try to explain these results by indicating infinite families of strongly regular graphs containing the above three graphs (or at least graphs with all properties mentioned above; in the cases Γ_0 and Γ_2 we have not checked whether the above data determine the graphs uniquely). All graphs found have known parameters, but some of them are not isomorphic to graphs found previously.

2. A variation on a theme by Calderbank and Kantor

It is well-known that there is a 1-1-1 correspondence between strongly regular graphs with regular elementary abelian automorphism group, projective 2-weight codes, and subsets X of a projective space such that $|X\cap H|$ takes only two values when H runs over all hyperplanes (cf. DELSARTE [4], CALDERBANK and KANTOR [3]). A survey of such structures can be found in [3]; BROUWER [1] constructs a few more. Here we give a variation on construction SU2 of [3] (see also KANTOR [7]).

Let $Q_1, \ldots, Q_c$ be a collection of pairwise disjoint nondegenerate quadrics of maximal Witt index in a projective space $PG(2m-1, q)$, let $M_1, \ldots, M_a$ and $N_1, \ldots, N_b$ be pairwise disjoint subspaces of projective dimension $m-1$, where each M_i is contained in some Q_j, and each N_i is disjoint from all Q_j. Put

$$X = (\bigcup_i N_i)\cup(\bigcup_i Q_i)\setminus(\bigcup_i M_i),$$

then

$$|X| = c\frac{(q^m-1)(q^{m-1}+1)}{q-1}+(b-a)\frac{q^m-1}{q-1} = e\cdot\frac{q^m-1}{q-1},$$

if we put

$$e = c(q^{m-1}+1)+b-a.$$

If H is a hyperplane, then $|X\cap H|=u$ in case H does not contain any of the N_i and either is not tangent to one of the Q_j, or contains some M_i, and $|X\cap H|=u+q^{m-1}$ otherwise, where

$$u = c\frac{q^{2m-2}-1}{q-1}+(b-a)\frac{q^{m-1}-1}{q-1} = e\frac{q^{m-1}-1}{q-1}.$$

(Note that two disjoint $(m-1)$-spaces span the entire projective space, so that a hyperplane H cannot be tangent to two Q_j simultaneously, or contain two M_i 's or N_i's.)

If we embed the projective space $PG(2m-1, q)$ as a hyperplane in a projective space $PG(2m, q)$ and construct a graph Γ with vertex set $PG(2m, q)\setminus PG(2m-1, q)$, where two vertices are adjacent when the line joining them hits X, then Γ will be strongly regular with parameters

$$v = q^{2m},$$
$$k = |X|(q-1) = e(q^m-1),$$
$$r = q^m-e,$$
$$s = -e$$

(where v is the number of vertices, k the valency, and k, r, s are the eigenvalues of Γ).

Thus, the graph Γ constructed will have the same parameters as the graphs constructed in [3] under SU2, starting from e pairwise disjoint $(m-1)$-spaces, but is not necessarily isomorphic to such a graph. (Note, e.g., that if m is odd, then a hyperbolic quadric Q does not have a spread — a partition into $(m-1)$-spaces.)

The graph Γ_0 is the special case of this construction where $q=2$, $m=4$, $a=c=1$, $b=0$. (It is a graph found under SU2, so its interest must be the fact that both subconstituents are strongly regular.)

3. A quadric with a hole

Let Q be a hyperbolic quadric in $PG(2m-1, q)$, and let M be a maximal totally isotropic (or totally singular) subspace of Q. Let Γ be the graph with vertex set Q, where two points are adjacent when they are orthogonal, i.e., when the line joining them is contained in Q. It is well known that Γ is strongly regular (in fact rank 3) with parameters

$$v_\Gamma = \frac{(q^m-1)(q^{m+1}+1)}{q-1},$$

$$k_\Gamma = q\,\frac{(q^{m-1}-1)(q^{m-2}+1)}{q-1},$$

$$\lambda_\Gamma = q^2\frac{(q^{m-2}-1)(q^{m-3}+1)}{q-1}+q-1,$$

$$\mu_\Gamma = \frac{(q^{m-1}-1)(q^{m-2}+1)}{q-1}$$

where λ (resp. μ) denotes the number of common neighbours of two adjacent (resp. nonadjacent) vertices.

Consider the graph Δ with point set $X=Q\setminus M$, where two points x, y are adjacent when the projective line xy is contained in X. (Thus, Δ is a partial subgraph, not an induced subgraph, of Γ.)

Define

$$\theta_j = \frac{q^j-1}{q-1}.$$

Theorem 1. *Δ is strongly regular with parameters $v=q^{m-1}\theta_m$, $k=q^{m-1}\theta_{m-1}$, $\lambda=q^{m-1}\theta_{m-2}+q^{m-2}(q-1)$, $\mu=q^{m-1}\theta_{m-2}$, $r=q^{m-1}$, $s=-q^{m-2}$. It has automorphism group $q^{m(m-1)/2}L_m(q)$, acting (imprimitively) rank 4 (for $m\geq 3$, $q>2$ or $m\geq 4$, $q=2$). The following three conditions are equivalent:* (i) *Δ is pairwise 4-regular,* (ii) *Δ is a graph in the switching class of a regular two-graph, and* (iii) $q=2$.

Proof. Compute the parameters of Δ as those of Γ minus what is missing. We find:

$$v = v_\Gamma - \theta_m,$$
$$k = k_\Gamma - q\theta_{m-1},$$
$$\lambda = \lambda_\Gamma - 2(\theta_{m-1}-1)+\theta_{m-2},$$
$$\mu = \mu_\Gamma - 2\theta_{m-1}+\theta_{m-2} = \lambda_\Gamma - \theta_m + 2.$$

[As an example, let us give details for the computation of μ. Let x, y be nonadjacent points. Then either the line xy is contained in Q (but hits M), or x, y are nonadjacent in Γ. In the first case, $x^\perp \cap M = y^\perp \cap M$, and the common neighbours of x and y in Δ are those in Γ except for the points in $\langle x, x^\perp \cap M\rangle \setminus \{x, y\}$, so that we find $\lambda_\Gamma - (\theta_m - 2)$ common neighbours. In the second case we have $x^\perp \cap M \neq y^\perp \cap M$ since $x^\perp \cap M$ is contained in only two maximal totally isotropic subspaces of Q; the number of common neighbours of x and y is μ_Γ minus $|x^\perp \cap \langle y, y^\perp \cap M\rangle| + |y^\perp \cap \langle x, x^\perp \cap M\rangle| = 2\theta_{m-1}$ plus $|x^\perp \cap y^\perp \cap M| = \theta_{m-2}$. (Note that x and y are the only isotropic points on the line xy, so that if z is a common neighbour of x and y in the graph Γ, then the plane $\langle x, y, z\rangle$ meets M in at most one point.)]

Aut Δ is transitive, and the stabilizer of a point x of Δ in Aut Δ has 4 orbits: $\{x\}$, $\Delta(x)$ (the set of neighbours of x in the graph Δ), $\Delta_{2A}(x)$ (the set of nonneighbours y of x such that $y^\perp \cap M = x^\perp \cap M$), and $\Delta_{2B}(x)$ (the set of remaining nonneighbours of x).

[Transitivity on each of these sets is clear, so we only have to check that the graph is not rank 3, i.e., that we can distinguish the suborbits $\Delta_{2A}(x)$ and $\Delta_{2B}(x)$. For $q>2$ and $m \geqq 3$ this is easy: if we write $A^\Delta = \bigcap_{a \in A} \Delta(a)$, then for nonneighbours x, y the set $\{x, y\}^{\Delta\Delta}$ equals $xy \cap X$, where xy denotes the projective line joining x and y, so that $y \in \Delta_{2A}(x)$ when this set has size q, and $y \in \Delta_{2B}(x)$ when it has size 2. If $q=2$ and $m=3$ then Δ is the triangular graph $T(8)$ and is in fact rank 3. If $m \geqq 4$, then we can recognize the maximal totally isotropic subspaces disjoint from M as cliques of size θ_m in Δ, and by taking intersections we find all totally isotropic subspaces disjoint from M. Now, given two nonadjacent vertices x, y let N_x be a maximal totally isotropic subspace on x disjoint from M; since we know the projective structure of N, we can find the subspace $\langle \Delta(y) \cap N\rangle$ of N. But $y \in \Delta_{2A}(x)$ precisely when $\langle \Delta(y) \cap N\rangle = (\Delta(y) \cap N) \cup \{x\}$.]

Concerning pairwise 4-regularity: each edge is in a constant number of 4-cliques, since Aut Δ acts edge-transitively. Remains to verify that each nonedge is in a constant number of $K_{1,1,2}$'s, i.e., that the number of edges in a μ-graph is constant. Consider two nonadjacent vertices x, y. If $y \in \Delta_{2A}(x)$, then $\Delta(x) \cap \Delta(y)$ is the graph with as vertex set a cone with as top the set $xy \setminus \{p\}$, if $xy \cap M = \{p\}$, raised on $Q_0 \setminus M_0$, where Q_0 is a hyperbolic quadratic in $PG(2m-5, q)$ and M_0 is a maximal totally isotropic subspace in Q_0, and where two vertices are adjacent when they project to adjacent vertices in Q_0 and the line joining them does not meet $\langle p, M_0\rangle$. Thus, $\Delta(x) \cap \Delta(y)$ is regular of valency

$$k_{\mu A} = q^2 \cdot q^{m-3}\theta_{m-3} + (q^2-q)((q-1)\theta_{m-3}+1) = q^{m-1}\theta_{m-3} + q(q-1)q^{m-3}.$$

If $y \in \Delta_{2B}(x)$, then $\Delta(x) \cap \Delta(y)$ is the graph with as vertex set the set $Q_1 \setminus (M_1 \cup M_2)$, where Q_1 is a hyperbolic quadric in $PG(2m-3, q)$ and M_1 and M_2 are maximal totally isotropic subspaces in Q_1 that meet in a subspace of codimension one in both, and where two vertices are adjacent when the line joining them is contained in $Q_1 \setminus (M_1 \cap M_2)$. Thus, $\Delta(x) \cap \Delta(y)$ is regular of valency

$$k_{\mu B} = k_{Q_1} - 2\theta_{m-2} - (q-2)\theta_{m-3} = q^{m-2}(\theta_{m-2}+1) - 2q^{m-3}.$$

We have $k_{\mu A} = k_{QB}$ if and only if $q=2$, and if that is the case, then

$$\mu = 2^{m-1}(2^{m-2}-1), \quad k_\mu = 2^{m-2}(2^{m-2}-1), \quad \beta = \frac{1}{2}\mu k_\mu = k_\mu^2. \quad \blacksquare$$

The graph Γ_1 is the special case of this construction where $q=2$, $m=4$. It is clear that Γ_1 is the first subconstituent of Γ_0. Remains to compare this construction with the description given in the introduction.

The folded halved 8-cube is the graph on the affine space 2^6 where two points are adjacent when the line joining them misses a hyperbolic quadric Q in the hyperplane at infinity. (Indeed, if $\mathbf{1}$ is the all-one vector in 2^8, then 2^6 may be identified with $\mathbf{1}^\perp \setminus \langle \mathbf{1} \rangle$, where $\perp$ is taken with respect to the standard inner product; and $\frac{1}{2}\, wt(x)$ defines a quadratic from.)

Thus, the 8-cocliques in the folded halved 8-cube are the affine 3-spaces in 2^6 with totally isotropic plane at infinity, and the two kinds of 8-cocliques correspond to the two kinds of totally isotropic planes on the hyperbolic quadric. Now the isomorphism with our graph Δ is stated by the following proposition.

Proposition 2. *The following three (strongly regular) graphs are isomorphic:*

(i) *the graph with vertex set $Q \setminus M$, Q a hyperbolic quadric in $PG(7, q)$, M a maximal totally isotropic subspace, where x and y are adjacent when the line xy is contained in $Q \setminus M$;*

(ii) *the graph defined in $PG(6, q)$ provided with a quadratic form of Witt index 4 with single point radical $\{m\}$, with vertex set the set of totally isotropic planes of one kind not containing m, where two planes are adjacent when they meet in a point;*

(iii) *the graph defined in the affine space $AG(6, q)$ where the hyperplane at infinity is provided with a hyperbolic quadric, with vertex set the collection of affine 3-spaces with totally isotropic plane of one kind at infinity, where two 3-spaces are adjacent when they have a line in common.*

Proof. For $m=4$ we can apply triality to the description of our graph Δ, and find that it is isomorphic to the graph of the totally isotropic 3-spaces of one kind not containing a fixed point m, where two 3-spaces are adjacent when they meet in a line. But there is a 1-1 correspondence between totally isotropic 3-spaces N not containing m and planes $N \cap m^\perp$, and we find description (ii).

Dualizing, our planes become 3-spaces, and m becomes a hyperplane carrying a quadratic from (if H is a hyperplane on m then we may set $Q(H) = Q(H^\perp)$), and we find description (iii). $\blacksquare$

4. Nonisotropic points plus a subspace

Now we have to look at Γ_2 and wonder why it is pairwise 4-regular. When $q=2$, then both the graph Γ on 2^{2m} vertices constructed in section 2 (for $e=2^{m-1}$) and its first subconstituent Δ constructed in section 3 are regular two-graphs, and by Theorem 1.3.3 of HAEMERS [5] (cf. BROUWER and VAN LINT [2], p. 111—112) we find that the second subconstituent E of Γ is strongly regular.

We have not yet thought about the pairwise 4-regularity of E — at first sight it seems as if one should distinguish cases, but no doubt there is some elegant argument.

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A. E. Brouwer

Department of Mathematics,
Techn. Univ. Eindhoven,
Eindhoven
The Netherlands

A. V. Ivanov and M. H. Klin

Inst. for System Studies,
Moscow,
Soviet Union